4 Limit of a Function at a Real Number a

4.1 The definition

Definition 4.1.1. A function f has the limit $L \in \mathbb{R}$ as x approaches a real number a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\big(a-\delta_0, a\big) \cup \big(a, a+\delta_0\big)$.

(II) For each real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < |x-a| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Remark 4.1.2. Notice that the condition that x belongs to the set $\big(a-\delta_0, a\big) \cup \big(a, a+\delta\big)$ can be expressed in terms of the distance between x and a as: $0 < |x-a| < \delta_0$.

The following figure illustrates Definition 4.1.1.

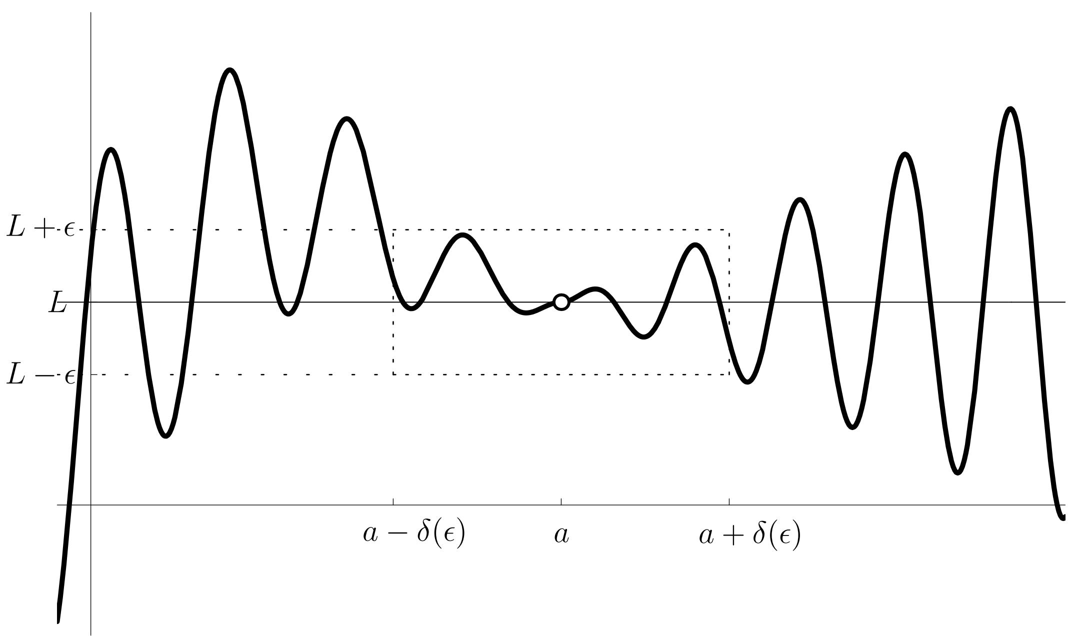
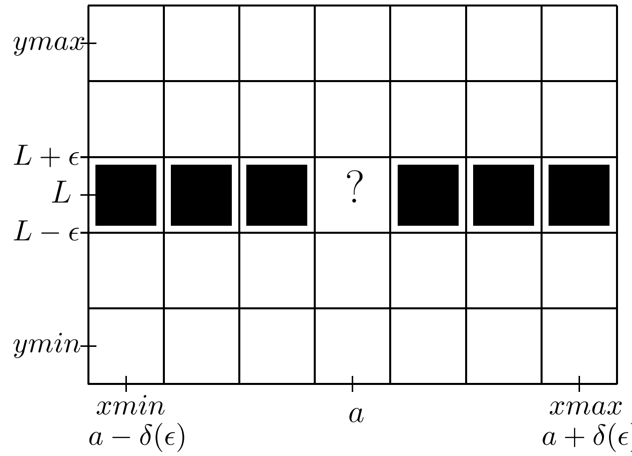


Figure 4:

Next we restate Definition 4.1.1 using the terminology of a calculator screen. The figure below shows a fictional calculator screen with 35 pixels. We assume that *ymin* and *ymax* are chosen in such a way that the number L is in the middle of the y-range and that *xmin* and *xmax* are such that a is in the middle of the x-range.

Definition 4.1.3 (Calculator Screen)**.** A function f has a limit L as x approaches a if (I) in Definition 4.1.1 is satisfied and

- for each choice of $ymin$ and $ymax$ there exists Δ (which depends on $ymin$ and $ymax$) such that $0 < \Delta \leq \delta_0$ and such that whenever we choose $xmin$ and $xmax$ such that $xmax - xmin < 2\Delta$ the graph of the function f will appear to be a straight horizontal line on the calculator screen with the only possible exception at the pixel containing $x = a$.



For the specific fictional calculator screen shown above, the connection between Definition 4.1.1 and Definition 4.1.3 is given by $\epsilon = (ymax - ymin)/8$, $xmin = a - \delta(\epsilon)$, $xmax = a + \delta(\epsilon)$ and $\delta(\epsilon) = \Delta$.

The fictional screen in the example below is chosen for its simplicity. The screen of TI-92 (see the manual p. 321) is 239 pixels wide and 103 pixels tall; it has 24617 pixels. The screen of TI-83 (see the manual p. 8-16) and of TI-82 is 95 pixels wide and 63 pixels tall; it has 5985 pixels. The screen of TI-85 (see the manual p. 4-13) is 127 pixels wide and 63 pixels tall; it has 8001 pixels. The screen of TI-89 (see the manual p. 222) is 159 pixels wide and 77 pixels tall; it has 12243 pixels. Using these numbers you can calculate the connection between ϵ and $\delta(\epsilon)$ in Definition 4.1.1 and the screen of your calculator.

4.2 Examples for Definition 4.1.1

Example 4.2.1. Prove $\lim_{x \to 2} (3x - 1) = 5$.

Solution. (I) Here $f(x) = 3x - 1$. This function is defined on $\mathbb{R}$. We can take any positive number for δ_0. Since it might be useful to have a specific δ_0 to work with, we set $\delta_0 = 1$.

Let $\epsilon > 0$ be given. Let $\delta(\epsilon) = \min\{\epsilon/3, 1\}$. Assume $0 < |x - 2| < \delta(\epsilon)$. Since $\delta(\epsilon) \leq \epsilon/3$, we conclude that $|x - 2| < \epsilon/3$. Next, we calculate

$$|(3x - 1) - 5| = |3x - 6| = 3\,|x - 2|\,. \tag{4.2.1}$$

It follows from the assumption $0 < |x - 2| < \delta(\epsilon)$ that $|x - 2| < \epsilon/3$. Therefore we conclude

$$|(3x - 1) - 5| = 3\,|x - 2| < 3\frac{\epsilon}{3} = \epsilon.$$

Thus we proved that

$$0 < |x - 2| < \delta(\epsilon) \quad \Rightarrow \quad |(3x - 1) - 5| < \epsilon.$$

This is exactly the implication in (II) in Definition 4.1.1. Since $\epsilon > 0$ was arbitrary this completes the proof. □

Remark 4.2.2. How did I guess the formula for $\delta(\epsilon)$ in the previous proof? I first studied the implication in the statement (II) in Definition 4.1.1. The goal in that implication is to prove

$$|(3x-1)-5| < \epsilon.$$

To prove this inequality we need to assume something about $|x-2|$. To find out what to assume, I simplified the expression $|(3x-1)-5|$ until $|x-2|$ appeared (see (4.2.1)). Then I solved for $|x-2|$. In this process of simplification I can afford to make the right-hand side larger. This will be illustrated in the next example.

Example 4.2.3. Prove $\lim_{x\to 2}(3x^2-2x-1) = 7$.

Solution. As usual, we first deal with (I). Again $f(x) = 3x^2-2x-1$ is defined on $\mathbb{R}$ and we can take any positive number for δ_0. Since it might be useful to have a specific choice of δ_0, we put $\delta_0 = 1$. (Notice that this implies that, from now on, we consider only in the values of x which are in the set $(1,2)\cup(2,3)$.)

Next we shall discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$|(3x^2-2x-1)-7| = |3x^2-2x-8| = |(3x+4)(x-2)| = |3x+4|\,|x-2|.$$

Now we use the fact that we are considering only the values of x which are in the set $(1,2)\cup(2,3)$. For $x\in(1,2)\cup(2,3)$ the value of $|3x+4|$ does not exceed 13. Therefore

$$|(3x^2-2x-1)-7| \leq 13\,|x-2| \quad \text{for all} \quad x\in(1,2)\cup(2,3). \tag{4.2.2}$$

Let $\epsilon > 0$ be given. The inequality $13\,|x-2| < \epsilon$ is easy to solve for $|x-2|$. The solution is $|x-2| < 3$ Now we define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{\epsilon}{13}, 1\right\}.$$

The remaining step of the proof is to prove the implication

$$|x-2| < \delta(\epsilon) \quad \Rightarrow \quad |(3x^2-2x-1)-7| < \epsilon.$$

I hope that at this point you can prove this implication on your own. ☐

Example 4.2.4. Prove $\lim_{x\to 2}\dfrac{x^3-x-4}{x-1} = 2$.

Solution. We first deal with (I). Notice that the function $f(x) = \dfrac{x^3-x-4}{x-1}$ is defined on $\mathbb{R}\setminus\{1\}$. In this proof we are interested in the values of x near $a=2$. Therefore, for δ_0 we can take any positive number which is smaller than 1. Since it is useful to have a specific number, we put $\delta_0 = 1/2$. (Notice that this implies that from now on we consider only the values of x which are in the set $(3/2,2)\cup(2,5/2)$.)

Next we shall discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\frac{x^3-x-4}{x-1} - 2\right| = \left|\frac{x^3-3x-2}{x-1}\right| = \left|\frac{(x^2+2x+1)(x-2)}{x-1}\right| = \left|\frac{x^2+2x+1}{x-1}\right|\,|x-2|. \tag{4.2.3}$$

Now remember that we are interested only in the values of x which are in the set $(3/2,2)\cup(2,5/2)$. For $x\in(3/2,2)\cup(2,5/2)$ we estimate

$$\left|\frac{x^2+2x+1}{x-1}\right| = \frac{x^2+2x+1}{x-1} \leq \frac{16}{1/2} = 32 \quad \text{for all} \quad x\in(3/2,2)\cup(2,5/2). \tag{4.2.4}$$

Combining (4.2.3) and (4.2.4) we get

$$\left|\frac{x^3-x-4}{x-1} - 2\right| \leq 32\,|x-2| \quad \text{for all} \quad x\in(3/2,2)\cup(2,5/2). \tag{4.2.5}$$

Let $\epsilon>0$ be given. The inequality $32\,|x-2|<\epsilon$ is very easy to solve for $|x-2|$. The solution is $|x-2|<\epsilon/32$. Now we define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{\epsilon}{32}, \frac{1}{2}\right\}.$$

The remaining piece of the proof is to prove the implication

$$|x-2|<\delta(\epsilon) \qquad \Rightarrow \qquad \left|\frac{x^3-x-4}{x-1} - 2\right| < \epsilon.$$

I hope that at this point you can prove this on your own. Write down all the details of your reasoning. □

Example 4.2.5. Prove $\lim\limits_{x\to 4}\sqrt{x}=2$.

Solution. As usual, we first deal with (I). Notice that the function $f(x)=\sqrt{x}$ is defined on $(0,+\infty)$. We are interested in the values of x near the point $a=4$. Thus, for δ_0 we can take any positive number which is <4. Since it is useful to have a specific number, we put $\delta_0=1$. (Notice that this implies that from now on in this proof we are interested only in the values of x which are in the set $(3,4)\cup(4,5)$.)

Next we shall discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\sqrt{x}-2\right| = \left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right| = \left|\frac{x-4}{\sqrt{x}+2}\right| = \left|\frac{1}{\sqrt{x}+2}\right|\,|x-4|. \tag{4.2.6}$$

Now remember that we are interested only in the values of x which are in the set $(3,4)\cup(4,5)$. For $x\in(3,4)\cup(4,5)$ we estimate

$$\left|\frac{1}{\sqrt{x}+2}\right| = \frac{1}{\sqrt{x}+2} \leq \frac{1}{\sqrt{3}+2} \leq \frac{1}{2} \quad \text{for all} \quad x\in(3,4)\cup(4,5). \tag{4.2.7}$$

Combining (4.2.6) and (4.2.7) we get

$$\left|\sqrt{x}-2\right| \leq \frac{1}{2}\,|x-4| \quad \text{for all} \quad x\in(3,4)\cup(4,5). \tag{4.2.8}$$

Let $\epsilon>0$ be given. The inequality $\frac{1}{2}\,|x-4|<\epsilon$ is easy to solve for $|x-4|$. The solution is $|x-4|<2\epsilon$. Now define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\{2\epsilon, 1\}.$$

The remaining step of the proof is to prove the implication

$$|x-4| < \min\{2\epsilon, 1\} \quad \Rightarrow \quad \left|\sqrt{x}-2\right| < \epsilon.$$

I hope that at this point you can prove this on your own. As before, please do it and write down the details of your reasoning. □

Example 4.2.6. Prove that for any $a > 0$, $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$.

Solution. Let $a > 0$. As before, we first deal with (I) in Definition 4.1.1. Notice that the function $f(x) = 1/x$ is defined on $\mathbb{R}\setminus\{1\}$. We are interested in the values of x near the point $a > 0$. Thus, for δ_0 we can take any positive number which is $< a$. Since it is useful to have a specific number, we put $\delta_0 = a/2$. (Notice that this implies that from now on in this proof we are interested only in the values of x which are in the set $(a/2, a) \cup (a, 3a/2)$.)

Next we shall discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{xa}\right| = \frac{|a-x|}{xa} = \frac{1}{xa}\,|x-a|\,. \tag{4.2.9}$$

Now remember that we are interested only in the values of x which are in the set $(a/2, a) \cup (a, 3a/2)$. For $x \in (a/2, a) \cup (a, 3a/2)$ we estimate

$$\frac{1}{xa} \leq \frac{1}{(a/2)a} = \frac{2}{a^2} \quad \text{for all} \quad x \in (a/2, a) \cup (a, 3a/2). \tag{4.2.10}$$

Combining (4.2.9) and (4.2.10) we get

$$\left|\frac{1}{x} - \frac{1}{a}\right| \leq \frac{2}{a^2}\,|x-a| \quad \text{for all} \quad x \in (a/2, a) \cup (a, 3a/2). \tag{4.2.11}$$

Let $\epsilon > 0$ be given. The inequality $\frac{2}{a^2}\,|x-a| < \epsilon$ is easy to solve for $|x-a|$. The solution is $|x-a| < (a^2/2)\epsilon$. Now define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{a^2}{2}\,\epsilon, \frac{a}{2}\right\}.$$

The remaining step of the proof is to prove the implication

$$|x-a| < \min\left\{\frac{a^2}{2}\,\epsilon, \frac{a}{2}\right\} \quad \Rightarrow \quad \left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon.$$

I hope that at this point you can prove this on your own. Write down the details of your reasoning. □

Exercise 4.2.7. Find each of the following limits. Prove your claims using Definition 4.1.1.

(a) $\lim_{x\to 3}(2x+1)$ (b) $\lim_{x\to 1}(-3x-7)$ (c) $\lim_{x\to 1}(4x^2+3)$

(d) $\lim_{x\to 2}\dfrac{x}{x-1}$ (e) $\lim_{x\to 3}\dfrac{x^2-x+2}{x+1}$ (f) $\lim_{x\to 0}x^{1/3}$

(g) $\lim_{x\to 0}\left(\dfrac{1}{|x|}\right)^{3/\ln|x|}$ (h) $\lim_{x\to 0}\tan x$ (i) $\lim_{x\to 0}\dfrac{1}{\cos x}$

(j) $\lim_{x\to 3}\dfrac{1}{x}$ (k) $\lim_{x\to 1}\dfrac{1}{x^2+1}$ (l) $\lim_{x\to -2}\dfrac{x}{x^2+4\,x+3}$

Exercise 4.2.8. Let $f(x)=\dfrac{x+1}{x^2-1}$. Does f have a limit at $a=1$? Justify your answer.

Exercise 4.2.9. Prove that for any $a>0$, $\lim_{x\to a}\sqrt{x}=\sqrt{a}$.

4.3 Infinite limits

Definition 4.3.1. A function f has the limit $+\infty$ as x approaches a real number a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0>0$ such that $f(x)$ is defined for each x in the set $\big(a-\delta_0,a\big)\cup\big(a,a+\delta_0\big)$.

(II) For each real number $M>0$ there exists a real number $\delta(M)$ such that $0<\delta(M)\le\delta_0$ and

$$0<|x-a|<\delta(\epsilon)\quad\Rightarrow\quad f(x)>M.$$

Definition 4.3.2. A function f has the limit $-\infty$ as x approaches a real number a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0>0$ such that $f(x)$ is defined for each x in the set $\big(a-\delta_0,a\big)\cup\big(a,a+\delta_0\big)$.

(II) For each real number $M<0$ there exists a real number $\delta(M)$ such that $0<\delta(M)\le\delta_0$ and

$$0<|x-a|<\delta(\epsilon)\quad\Rightarrow\quad f(x)<M.$$

Exercise 4.3.3. Find each of the following limits. Prove your claims using the appropriate definition.

(a) $\lim_{x\to 0}\dfrac{1}{|x|}$ (b) $\lim_{x\to -3}\dfrac{1}{(x+3)^2}$ (c) $\lim_{x\to 2}\dfrac{x-3}{x(x-2)^2}$

(d) $\lim_{x\to -1}\dfrac{x}{(x+1)^4}$ (e) $\lim_{x\to +\infty}\dfrac{x^2-x+2}{x+1}$ (f) $\lim_{x\to +\infty}\dfrac{x^2-x}{3-x}$